UNIT - IV

Central force:

When a particle is subject to the action of a force which is always either towards or away from a fixed point, the particle is said to be under the action of a central force.

* The velocity components in the radial and transverse directions are $\dot{r}, r\dot{\theta}$

* The acceleration components in the radial and transverse directions are $\ddot{r} - r\dot{\theta}^2, \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

Note:

polar equation of conic is $\frac{l}{r} = 1 + e\cos\theta$

Equiangular spiral:

Equiangular spiral is a curve which is such that the angle between the radius vector and the respective tangent is a constant angle say α

It's polar equation is $r = Ae^{(\cot\alpha)\theta}$

where A is constant.

Problems:

1. The velocity of the particle along and perpendicular to the radius vector are λr and $\mu\theta$. find the path and S.T the acceleration components along and perpendicular to the radius vector are

$$\lambda^2 r - \frac{\mu^2\theta^2}{r}, \quad \mu\theta\left(\lambda + \mu/r\right).$$

Soln:

We know that velocity components are $\dot{r}, r\dot{\theta}$

given $\dot{r} = \lambda r \Rightarrow \frac{dr}{dt} = \lambda r \longrightarrow (1)$

$$r\dot{\theta} = \mu\theta \Rightarrow r\frac{d\theta}{dt} = \mu\theta \longrightarrow (2)$$

$$\dot{\theta} = \frac{\mu\theta}{r}$$

(1) ÷ (2)

$$\frac{dr}{dt} \times \frac{dt}{r\,d\theta} = \frac{\lambda r}{\mu\theta}$$

$$\frac{dr}{r\,d\theta} = \frac{\lambda r}{\mu\theta}$$

$$\frac{1}{r^2}\,dr = \frac{\lambda}{\mu}\,\frac{d\theta}{\theta}$$

$$\int r^{-2}\,dr = \frac{\lambda}{\mu}\int\frac{d\theta}{\theta}$$

$$\frac{r^{-1}}{-1} = \frac{\lambda}{\mu}\log\theta + c$$

$$\frac{-1}{r} = \frac{\lambda}{\mu}\log\theta + c$$

Hence required equation of path is $-\frac{1}{r} = \frac{\lambda}{\mu}\log\theta + c$

Acceleration Components are,

$$\ddot{r} - r\dot{\theta}^2 \text{ and } \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

$$\ddot{r} - r\dot{\theta}^2 = \frac{d}{dt}(\dot{r}) - r\dot{\theta}^2$$

$$= \frac{d}{dt}(\lambda r) - r\left(\frac{\mu}{r}\theta\right)^2$$

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$$= \lambda \frac{dr}{dt} - r\frac{\mu^2}{r^2}\theta^2$$

$$= \lambda \frac{dr}{dt} - \frac{\mu^2\theta^2}{r}$$

$$= \lambda(\dot{r}) - \frac{\mu^2\theta^2}{r}$$

$$= \lambda(\lambda r) - \frac{\mu^2\theta^2}{r}$$

$$\ddot{r} - r\dot{\theta}^2 = \lambda^2 r - \frac{\mu^2\theta^2}{r}$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2 . \frac{\mu}{r} . \theta)$$

$$= \frac{1}{r}\frac{d}{dt}(\mu r\theta)$$

$$= \frac{1}{r}\mu\frac{d}{dt}(r\theta)$$

$$= \frac{1}{r}\mu(r\dot{\theta} + \theta\dot{r})$$

$$= \frac{\mu}{r} \cdot (r(\mu/r)\theta + \theta . \lambda r)$$

$$= \frac{\mu}{r} \cdot (\mu\theta + \theta\lambda r)$$

$$= \frac{\mu\theta}{r} \cdot (\mu + \lambda r)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \mu\theta\left(\frac{\mu}{r} + \lambda\right)$$

2. The velocities of the particle along and $\perp r$ to the radius vector from a fixed origin are a and b. find the path and the acceleration along and $\perp r$ to the radius vector.

Soln:

Velocity Components are $\dot{r}$ and $r\dot{\theta}$

given $\dot{r} = a$, $r\dot{\theta} = b$

$$\dot{\theta} = b/r$$

$$\frac{dr}{dt} = a \longrightarrow ①$$

$$r\frac{d\theta}{dt} = b \longrightarrow ②$$

① ÷ ②

$$\frac{dr}{dt} \cdot \frac{dt}{r\,d\theta} = \frac{a}{b}$$

$$\frac{dr}{r} = \frac{a}{b}\,d\theta$$

$$\int \frac{dr}{r} = \frac{a}{b}\int d\theta$$

$$\log r = \frac{a}{b}\theta + \log c$$

$$\log r - \log c = \frac{a}{b}\theta$$

$$\log(r/c) = \frac{a}{b}\theta$$

$$\frac{r}{c} = e^{(a/b)\theta}$$

$$r = c\,e^{(a\theta/b)}$$

∴ The path is equiangular spiral.

Acceleration Components are $\ddot{r} - r\dot{\theta}^2$ and $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$$\ddot{r} - r\dot{\theta}^2 = \frac{d}{dt}(\dot{r}) - r\left(\frac{b}{r}\right)^2$$

$$= \frac{d}{dt}(a) - r\frac{b^2}{r^2}$$

$$= 0 - \frac{b^2}{r}$$

$$\ddot{r} - r\dot{\theta}^2 = -b^2/r$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}(r^2 b/r)$$

$$= \frac{1}{r}\frac{d}{dt}(br)$$

$$= \frac{b}{r}\dot{r}$$

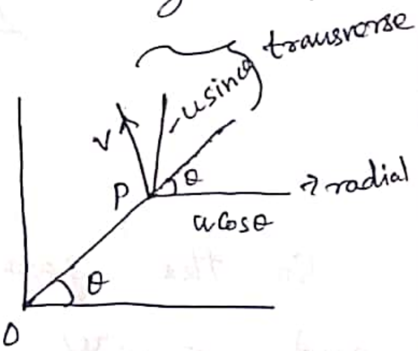
$$= \frac{b}{r}a$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{ab}{r}.$$

3. S.T the path of the point 'p' whose velocity is such that its a components in a fixed direction and in the direction $\perp r$ to the line joining 'P' to a fixed point 'O' are respectively the Constant u and v is a Conic with a focus at O and the eccentricity $\frac{u}{v}$.

Soln:

Let O be the pole and OA the given fixed direction, the initial line.



The Components of velocity 'u' in the radial and transverse directions are $u\cos\theta$, $-v\sin\theta$.

radial velocity $= \dot{r} = u\cos\theta$

transverse velocity $= r\dot{\theta} = v + (-u\sin\theta) = v - u\sin\theta$

$$\frac{dr}{dt} = u\cos\theta \longrightarrow (1)$$

$$r\frac{d\theta}{dt} = v - u\sin\theta \longrightarrow (2)$$

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$$\frac{①}{②} \Rightarrow \frac{dr}{dt} \times \frac{dt}{r d\theta} = \frac{u\cos\theta}{v - u\sin\theta}$$

$$\frac{dr}{r} = \frac{u\cos\theta}{v - u\sin\theta} d\theta$$

$$\int \frac{dr}{r} = \int \frac{u\cos\theta}{v - u\sin\theta} d\theta$$

$$\log r = -\log(v - u\sin\theta) + \log c$$

$$\log r = \log \frac{c}{v - u\sin\theta}$$

$$r = \frac{c}{v - u\sin\theta}$$

$$\frac{1}{r} = \frac{v - u\sin\theta}{c}$$

$$\frac{c}{r} = v - u\sin\theta$$

$$\frac{c/v}{r} = 1 - \frac{u}{v}\sin\theta$$

$$e = \frac{u}{v}$$

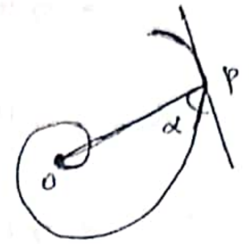
So the path is a Conic with its focus at 'O' and $e = u/v$.

4. A point 'p' describes and equiangular spiral with a constant angular velocity about the pole 'O'. S.T its acceleration varies as op and is in a direction making with the tangent at p with

the same Constant angle that op makes,

Soln:

Let α be a Constant angle



The equation of equiangular spiral is $r = a e^{(\cot\alpha)\theta}$

The angular velocity $= \dot{\theta} = \omega$

$$r = a e^{(\cot\alpha)\theta}$$

$$\dot{r} = a e^{(\cot\alpha)\theta} \cdot \cot\alpha\,\dot{\theta}$$

radial velocity $\dot{r} = r\omega \cot\alpha$

$$\ddot{r} = \dot{r}\,\omega \cdot \cot\alpha$$

$$= (r\omega \cot\alpha)\omega \cot\alpha$$

$$\ddot{r} = r\omega^2 \cot^2\alpha .$$

radial acceleration Component $= \ddot{r} - r\dot{\theta}^2$

$$= r\omega^2 \cot^2\alpha - r\omega^2$$

$$= r\omega^2 \cdot (\cot^2\alpha - 1)$$

Transverse acceleration Component $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$$= \frac{1}{r}\frac{d}{dt}(r^2\omega)$$

$$= \frac{\omega}{r}\frac{d}{dt}(r^2)$$

$$= \frac{\omega}{r}(2r\dot{r})$$

$$= 2\omega(r\omega\cot\alpha)$$

$$= 2\omega^2 r \cot\alpha$$

$$\text{Acceleration} = \sqrt{(r\omega^2(\cot^2\alpha - 1))^2 + (2\omega^2 r \cot\alpha)^2}$$

$$= r\omega^2 \sqrt{\cot^4\alpha - 2\cot^2\alpha + 1 + 4\cot^2\alpha}$$

$$= r\omega^2 \sqrt{\cot^4\alpha + 2\cot^2\alpha + 1}$$

$$= r\omega^2 \sqrt{(1 + \cot^2\alpha)^2}$$

$$= r\omega^2 (1 + \cot^2\alpha)$$

$$= r\omega^2 \operatorname{cosec}^2\alpha$$

$\therefore$ Acceleration is $\propto r$.

If β is the angle between radial directions, transverse direction of acceleration

$$\tan\beta = \frac{\text{Transverse Component}}{\text{radial Component}}$$

$$= \frac{2\omega^2 r \cot\alpha}{\omega^2 r (\cot^2\alpha - 1)}$$

$$= \frac{2\cot\alpha}{(\cot^2\alpha - 1)}$$

$$= \frac{2 \cdot \frac{1}{\tan\alpha}}{\left(\frac{1}{\tan^2\alpha} - 1\right)} = \frac{2/\tan\alpha}{\frac{1 - \tan^2\alpha}{\tan^2\alpha}}$$

$$= \frac{2}{\tan\alpha} \times \frac{\tan^2\alpha}{1 - \tan^2\alpha}$$

$$\tan\beta = \frac{2\tan\alpha}{1 - \tan^2\alpha}$$

$$\tan\beta = \tan 2\alpha$$

$$\boxed{\beta = 2\alpha}$$

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Central force - Central orbits

Def:

A particle p which is subjected to a force with acts always along the line joining p and a fixed point o. is said to move under a central force. The path traced out by the particle is called central orbit. The force is called attractive or replusive according as it is from p towards 'o' or from 'o' towards P.

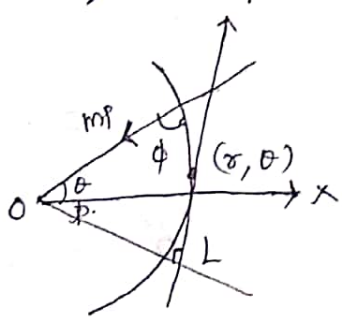
The fixed point 'o' is called the central of force.

Book work:

Differential equation to a central orbit

Let mp be the central force acting on a particle whose position is represented by (r,θ).

Let the force be towards the fixed point o, the pole.



The equation of motion are

$$m(\ddot{r}-r\dot{\theta}^2) = -mp \longrightarrow (1)$$

and

$$\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \longrightarrow (2)$$

from ② we get $r^2\dot{\theta} = $ Constant

$$r^2\dot{\theta} = h$$

$$\dot{\theta} = \frac{h}{r^2}$$

put $u = \frac{1}{r}$

$$\dot{\theta} = hu^2 \longrightarrow ③$$

$$r = \frac{1}{u}$$

$$\dot{r} = -\frac{1}{u^2}\frac{du}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= -\frac{1}{u^2}\frac{du}{d\theta} \times hu^2$$

$$\dot{r} = -h\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$\ddot{r} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$= -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}hu^2$$

$$\ddot{r} = -h^2u^2\frac{d^2u}{d\theta^2}$$

Sub these values in ① we obtain

$$-h^2u^2 \frac{d^2u}{d\theta^2} - \frac{1}{u} h^2u^4 = -P$$

$$h^2u^2 \cdot \left(\frac{d^2u}{d\theta^2} + u \right) = P$$

(or)

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$$

This is the differential equation to a Central orbit. Integrating this subject to given initial conditions gives (u, θ) equation to the Central Orbit.

1. To find the differential equation of a Central orbit in pedal (p, r) Coordinates where p is the $\perp r$ distance from the pole on the tangent.

Soln:

W.K.T

$$p = r\sin\phi$$

$$\frac{1}{p^2} = \frac{1}{r^2} \operatorname{Cosec}^2\phi$$

$$= \frac{1}{r^2} [1 + \operatorname{Cot}^2\phi]$$

But, $\tan\phi = r \frac{d\theta}{dr}$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} \left[1 + \left(\frac{dr}{r d\theta} \right)^2 \right]$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta} \right)^2 \longrightarrow (1)$$

put $r = \frac{1}{u}$

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

$$① \Rightarrow \frac{1}{p^2} = u^2 + \frac{1}{r^4}\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = u^2 + \frac{r^4}{r^4}\left(\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

Diff w.r.t 'r'

$$-\frac{2}{p^3}\frac{dp}{dr} = 2u\frac{du}{dr} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dr}$$

$$-\frac{2}{p^3}\cdot\frac{dp}{dr} = 2u(-u^2) + 2\frac{d^2u}{d\theta^2}(-u^2)$$

ie), $\frac{1}{p^3}\frac{dp}{dr} = u^2\left(\frac{d^2u}{d\theta^2} + u\right)$

But $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}$

$$\therefore \frac{1}{p^3}\frac{dp}{dr} = u^2\cdot\frac{P}{h^2u^2}$$

(or)

$$P = \frac{h^2}{p^3}\frac{dp}{dr}.$$

This gives the differential equation to the central orbit in (p-r) form.

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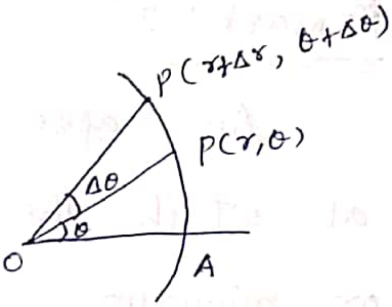
Book work:

Areal velocity of a moving particle p.

Soln:

Let $P(r,\theta)$ be the position of the particle at any time t and $Q(r+\Delta r; \theta+\Delta\theta)$ its position at time $t+\Delta t$



Area of the sector POQ

$$\Delta A = OA = \text{area of } \Delta POQ$$

$$= \frac{1}{2} OP \cdot OQ \sin \angle POQ$$

$$= \frac{1}{2} r(r+\Delta r) \sin \Delta\theta$$

$$= \frac{1}{2} r^2 \Delta\theta$$

$$\frac{dA}{dt} = \lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h$$

Remark : 1

The speed v of the particle is given by

$$v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

$$v^2 = \left(-h \frac{du}{d\theta}\right)^2 + \frac{1}{u^2} h^2 u^4$$

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$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$$v^2 = \frac{h^2}{p^2}$$

$$\boxed{h = pv}$$

Remark : 2

An apse in a Central orbit, is a point at which the radius vector is a maximum or minimum. ie), r is maximum (or) minimum Hence u is minimum (or) maximum.

$$\therefore \frac{du}{d\theta} = 0.$$

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2, \quad u^2 = \frac{1}{r^2}$$

$\therefore$ At an apse point $p = r$ so that the direction of motion is perpendicular to the radius vector. The radius vector is called apse line. The Corresponding value of r is called "Apsidal distance".